Semigroup approach to conservation laws with discontinuous flux

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Abstract

The model one-dimensional conservation law with discontinuous *spatially heterogeneous* flux is

$$u_t + \mathfrak{f}(x,u)_x = 0, \quad \mathfrak{f}(x,\cdot) = f^l(x,\cdot)1\!\!1_{x<0} + f^r(x,\cdot)1\!\!1_{x>0}. \tag{EvPb}$$

We prove well-posedness for the Cauchy problem for (EvPb) in the framework of solutions satisfying the so-called adapted entropy inequalities.

Exploiting the notion of integral solution that comes from the nonlinear semigroup theory, we propose a way to circumvent the use of strong interface traces for the evolution problem (EvPb) (in fact, proving existence of such traces for the case of x-dependent $f^{l,r}$ would be a delicate technical issue). The difficulty is shifted to the study of the associated one-dimensional stationary problem $u+\mathfrak{f}(x,u)_x = g$, where existence of strong interface traces of entropy solutions is an easy fact. We give a direct proof of this fact, avoiding the subtle arguments of kinetic formulation [23] or of the H-measure approach [27].

1 Introduction

Scalar conservation law with space-discontinuous flux was a subject of intense study since twenty years. The goal of this note is to highlight the results that can be inferred from the nonlinear semigroup approach (see [12, 14]) to such problems, specifically for the case of space dimension one.

We stick to the unifying framework for proving existence, uniqueness, stability, convergence of numerical approximations that was proposed in the paper [7] of K.H. Karlsen, N.H. Risebro and the author. In [7], we have studied the model problem

$$u_t + \mathfrak{f}(x,u)_x = 0, \quad \mathfrak{f}(x,\cdot) = f^l(x,\cdot)1\!\!1_{x<0} + f^r(x,\cdot)1\!\!1_{x>0} \tag{EvPb}$$

under the *space homogeneity* assumption $f^{l,r}(x,\cdot) \equiv f^{l,r}(\cdot)$. This assumption appears as a technical one, nevertheless it was a cornerstone of the entropy formulation because of the explicit use of *strong interface traces* within the uniqueness technique of [7]. Presently, to the authors' knowledge there is no proof of existence of strong traces for the non-homogeneous case. And even though such result is expected to be true under some weak assumptions on the dependence of $f^{l,r}$ on u and x, the proof (following the well-established kinetic techniques [29, 23] or H-measure techniques [26, 27]) would be rather lengthy and highly technical. The semigroup approach exploited in the present note permits us to circumvent the difficulty, for the one-dimensional case. Actually, we will justify existence of strong interface traces in a particularly simple setting, using the least technical part of the ideas of [27]. Then we will conduct a brief study of the operator governing (EvPb) and apply general principles of the nonlinear semigroup theory.

Let us recall the main features of the entropy formulation of Karlsen, Riesbro and the author [7] for the case $f^{l,r}(x,u) \equiv f^{l,r}(u)$. We postulated that a function $u \in L^\infty((0,T)\times\mathbb{R})$ is a $\mathcal{G}$-entropy solution of (EvPb) if

(i) It is an entropy solution in the classical sense of Kruzhkov [22] away from the interface $\{x=0\}$, i.e., in the subdomains $\Omega^l := (0,T)\times\mathbb{R}^-$ and $\Omega^r := (0,T)\times R^+$;

(ii) Moreover, the two solutions are coupled across the interface $\{x=0\}$ by the relation

$$\big(\gamma^l u, \gamma^r u\big)(t) \in \mathcal{G} \quad \text{for a.e. } t \in (0,T). \tag{1.1}$$

Here $\gamma^l u, \gamma^r u$ are strong (in the L^1 sense) traces of local entropy solutions $u|_{\Omega^l}$ and $u|_{\Omega^r}$, respectively: see [27] (and also [23]) for the proof of existence of these traces in the homogeneous case[1]. Further, $\mathcal{G} \subset \mathbb{R}^2$ is an L^1*-dissipative germ*, that is, a set of couples (u^l, u^r) encoding the Rankine-Hugoniot (conservativity) condition

$$\forall (u^l, u^r) \in \mathcal{G} \qquad f^l(u^l) = f^r(u^r) \tag{1.2}$$

and the interface dissipation condition

$$\forall (u^l, u^r), (c^l, c^r) \in \mathcal{G} \qquad q^l(u^l, c^l) \geq q^r(u^r, c^r) \tag{1.3}$$

with $q^{l,r}$ the Kruzhkov entropy fluxes given by

$$q^{l,r}(\cdot, c) = \operatorname{sign}(\cdot - c)\big(f^{l,r}(\cdot) - f^{l,r}(c)\big). \tag{1.4}$$

Further, [7] provides a *global entropy formulation* (see Definition 2.3 below) which is shown to be equivalent to (ii) whenever the one-sided traces $\gamma^{l,r} u$ on $\{x = 0\}$ do exist. Yet the global entropy formulation avoids the explicit use of interface traces (such as (1.1) above); for this reason, it is especially useful for proving existence of solutions and convergence of various approximation procedures. Our goal is to provide a uniqueness proof that relies on this global entropy formulation. To this end, we combine two ideas.

Firstly, we observe that one can use the technique of the "comparison" proof of [7, Th.3.28] in the case where one works with solutions u and $\hat{u}$ such that *only one of them* (say, $\hat{u}$) has strong interface traces. In this paper, we will say that $\hat{u}$ is *trace-regular* if $\gamma^l \hat{u}$ and $\gamma^r \hat{u}$ exist in the sense of Definition 2.1 below.

Thus, we are able to "compare" a general solution and a trace-regular solution. Here the second ingredient comes into play. Indeed, the trace-regularity issue is particularly simple in the one-dimensional case for the so-called *stationary problem*:

$$u + \mathfrak{f}(x, u)_x = g \tag{StPb}$$

where $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and a $\mathcal{G}$-entropy solution of (StPb) is sought for. In Lemma 3.1 we give a trace-regularity result based on elementary arguments. Now, problem (StPb) can be seen as the resolvent equation

$$u + A_{\mathcal{G}} u \ni g \tag{AbSt}$$

associated with the abstract evolution equation

$$\frac{d}{dt} u + A_{\mathcal{G}} u \ni h, \quad u(0) = u_0. \tag{AbEv}$$

Here $A_{\mathcal{G}}$ is the operator $u \mapsto \mathfrak{f}(x, u)_x$ defined on the appropriate domain $D(A_{\mathcal{G}}) \subset L^1(\mathbb{R})$ by its graph: $A_{\mathcal{G}} = \{(u, z) \in (L^1(\mathbb{R}))^2 \mid z \in A_{\mathcal{G}} u\}$. As a matter of fact, we will require that $u \in D(A_{\mathcal{G}})$ be trace-regular functions. Then the notion of integral solution can be exploited, following [12, 14], as it was done in [16, 3, 6] in various contexts. Indeed, u is an integral solution of (AbEv) if the comparison inequality in $\mathcal{D}'(0, T)$ holds:

$$\forall (\hat{u}, z) \in A_{\mathcal{G}} \qquad \frac{d}{dt} \|u(t) - \hat{u}\|_{L^1} \leq \big[u(t) - \hat{u}, h - z\big]_{L^1} \tag{1.5}$$

where the right-hand side is the so-called L^1 bracket (see Definition 3.6 below). Notice that within the semigroup approach, we limit our attention to $L^1 \cap L^\infty$ data (see Corollary 2.8 and Section 5 for a generalization to L^∞ data, which is not trivial).

Here is our point:

property (1.5) *(with $z = g - u$) can be established*
whenever u is a $\mathcal{G}$-entropy solution of (EvPb)
and $\hat{u}$ is a trace-regular $\mathcal{G}$-entropy solution of (StPb).

This observation closes the loop, because we deduce uniqueness of a $\mathcal{G}$-entropy solution to the evolution problem from the uniqueness of the integral solution. The latter uniqueness comes for gratis from the general principles of the nonlinear semigroup theory as soon as we prove that $A_{\mathcal{G}}$ is a densely defined accretive operator on $L^1(\mathbb{R})$ with m-accretive closure.

[1] Actually, a non-degeneracy of $f^{l,r}$ on intervals is needed for existence of such traces, see assumption (H3). But if the degeneracy happens, one can reformulate (1.1) in terms of the traces of some "singular mapping functions" $V f^{l,r}(u)$, see [7].

The paper is organized as follows. In Section 2 we state the assumptions, definitions and the main result. In Section 3 we study the stationary problem (StPb) and establish the main properties of the operator $A_{\mathcal{G}}$ on $L^1(\mathbb{R})$ associated with the formal expression $u \mapsto \mathfrak{f}(x,u)_x$. In particular, we show that the domain of $A_{\mathcal{G}}$ can be restricted to trace-regular functions. Then in Section 4 we deduce the uniqueness in the setting of $\mathcal{G}$-entropy solutions for problem (EvPb) with $L^1 \cap L^\infty$ data. Finally, in Section 5 we discuss the application of the idea of this paper for the one-dimensional Dirichlet boundary-value problem for conservation law; we also treat the case of merely L^∞ data for problem (EvPb). The Appendix of the paper contains a technical result on entropy solutions of a spatially non-homogeneous conservation law; this result has some interest on its own.

2 Assumptions, definitions and results

Let us denote $\mathbb{R}^l := (-\infty, 0)$ and $\mathbb{R}^r := (0, +\infty)$, so that $\Omega^{l,r} = (0,T) \times \mathbb{R}^{l,r}$. For the sake of simplicity of the presentation, let us assume

$$\forall x \in \mathbb{R}^{l,r} \text{ the functions } u \mapsto f^{l,r}(x,u) \text{ are supported in } [0,1]. \tag{H1}$$

This assumption is only used to ensure a uniform L^∞ bound on solutions and on approximate solutions[2]. For the sake of generality we will consider $\mathbb{R}$-valued bounded functions u_0 and g, although (H1) naturally appears in the case where solutions are $[0,1]$-valued (such solutions represent saturations in the porous media, sedimentation or road traffic models; see, e.g., [1, 17, 5]).

Throughout this paper, we assume that $f^{l,r}$ verify

$$\begin{gathered} f^{l,r} \text{ are Lipschitz continuous in } (x,u) \in \mathbb{R}^{l,r} \times [0,1], \\ \text{and } f^{l,r}(0,\cdot) \text{ have a finite number of extrema on } [0,1]. \end{gathered} \tag{H2}$$

We will also require the genuine nonlinearity property:

$$\forall x \in \mathbb{R}^{l,r} \text{ the functions } u \mapsto (f^{l,r})_u(x,u) \text{ do not vanish on subintervals of } [0,1]. \tag{H3}$$

Notice that these assumptions can be relaxed but we stick to the above hypotheses for the sake of simplicity.

Let us give the main definitions. Firstly, we recall the notion of strong boundary trace for the case of the domain $(0,T) \times \mathbb{R}^l$ (the case of $(0,T) \times \mathbb{R}^r$ is analogous)[3]. What is needed for our case is:

Definition 2.1 *Let $u \in L^\infty((0,T) \times (-\infty,0))$. Then $\gamma^l u \in L^\infty(0,T)$ is the strong trace of u on the boundary $\{x=0\} := \{(t,0) \,|\, t \in (0,T)\}$ if $u(\cdot,x)$ converges to $(\gamma^l u)(\cdot)$ essentially in $L^1(0,T)$ as $x \uparrow 0$.*

Next, we define *germs* in terms of fluxes $f^{l,r}$ corresponding to the "frozen" value $x=0$. Prescribing a *complete, maximal* L^1D-germ is a way to prescribe the interface coupling at $\{x=0\}$ (see [7]).

Definition 2.2 (L^1-dissipative germs) *A subset $\mathcal{G}$ of $\mathbb{R}^2$ is called L^1D-germ (germ, for short) if it satisfies* (1.2) *and* (1.3) *with the fluxes $f^{l,r}$ evaluated at $x=0$.*

Such a germ is called maximal if it possesses no non-trivial extension; it is called definite if it possesses only one maximal extension, in which case the extension is denoted by $\mathcal{G}^*$. Finally, it is called complete if any Riemann problem for the auxiliary conservation law

$$u_t + \left(f^l(0,u)\mathbb{1}_{x<0} + f^r(0,u)\mathbb{1}_{x>0}\right)_x = 0 \tag{2.1}$$

admits a solution satisfying (i),(ii) in the Introduction.

The completeness means that for any $(u_-, u_+) \in \mathbb{R}^2$ there exists a couple $(c^l, c^r) \in \mathcal{G}$ such that u_- can be joined to c^l by a Kruzhkov-admissible wave fan with negative speed for the flux $f^l(0,\cdot)$ and c^r can be joined to u_+ by a Kruzhkov-admissible wave fan with positive speed for the flux $f^r(0,\cdot)$. Notice that in this case, the so constructed function u is self-similar, therefore it possesses interface traces (in the strong sense of $L^1(0,T)$ convergence of $u(r,\cdot) \to (\gamma^r u)(\cdot)$ and of $u(-r,\cdot) \to (\gamma^l u)(\cdot)$ as $r \to 0^+$) that verify $\gamma^{l,r} u = c^{l,r}$.

The following definition (cf. [10, 9, 17, 7]), however, avoids the explicit reference to the point (ii) of the introduction.

[2] see [7] for more general assumptions that ensure L^∞ bounds, that have to be adapted to the inhomogeneous case.

[3] For the multi-dimensional domains treated in the Appendix, one uses an analogous definition based upon a parametrizaton of a neighbourhood of $\partial\Omega$ by $(\sigma, h) \in \partial\Omega \times (0,1)$.

Definition 2.3 ($\mathcal{G}$-entropy solution of the evolution problem)
Assume we are given an L^1D-germ $\mathcal{G}$. A function $u \in L^\infty((0,T)\times\mathbb{R})$ is called $\mathcal{G}$-entropy solution of (EvPb) *with an initial datum $u(0,\cdot) = u_0 \in L^\infty(\mathbb{R})$ if it satisfies the Kruzhkov entropy inequalities away from the interface $\{x=0\}$:*

$$\forall c \in \mathbb{R}\ \ |u-c|_t + \mathrm{sign}(u-c)\mathfrak{f}_x(x,c) + \mathfrak{q}(x;u,c)_x \le 0,\ \ |u-c|\big|_{t=0} = |u_0 - c|\ \ in\ \mathcal{D}'\big([0,T)\times(\mathbb{R}\setminus\{0\})\big) \tag{2.2}$$

and if, in addition, it satisfies the global adapted entropy inequalities

$$|u-c(x)|_t + \mathrm{sign}(u-c(x))\mathfrak{f}_x(x,c(x)) + \mathfrak{q}(x;u,c(x))_x \le 0\ \ in\ \mathcal{D}'((0,T)\times\mathbb{R}) \tag{2.3}$$

for every function $c(\cdot)$ of the form

$$c(x) = c^l\,\mathbb{1}_{x<0} + c^r\,\mathbb{1}_{x>0}\quad with\ (c^l,c^r)\in\mathcal{G}^*. \tag{2.4}$$

In the inequalities (2.2),(2.3) the Kruzhkov entropy flux $\mathfrak{q} = q^l\,\mathbb{1}_{x<0} + q^r\,\mathbb{1}_{x>0}$ is computed with the help of (1.4), with the tacit x-dependency in $f^{l,r}$. Notice that with respect to the case of spatially homogeneous $f^{l,r}$, there is the additional term $\mathfrak{f}_x(x,c(x))$; the notation $\mathfrak{f}_x(x,c(x))$ ignores the discontinuity at zero, i.e.,

$$\mathfrak{f}_x(x,c(x)) := f^l_x(x,c^l)\mathbb{1}_{x<0} + f^r_x(x,c^r)\mathbb{1}_{x>0}.$$

Remark 2.4 *Note that it can be assumed, without loss of restriction, that a $\mathcal{G}$-entropy solution u belongs to $C([0,T];L^1_{loc}(\mathbb{R}))$. This is a consequence of the Kruzhkov inequalities in domains $\Omega^{l,r}$; see, e.g., [26, 4, 18] and references therein. In the sequel, we will always select the time-continuous representative of u; in particular, the initial condition can be taken in the sense $u(0,\cdot) = u_0$.*

The definition for the stationary problem (StPb) is analogous, cf. [15].

Definition 2.5 ($\mathcal{G}$-entropy solution of the stationary problem)
Assume we are given an L^1D-germ $\mathcal{G}$. A function $u \in L^\infty(\mathbb{R})$ is called $\mathcal{G}$-entropy solution of (StPb) *if it satisfies the Kruzhkov entropy inequalities*

$$\forall c \in \mathbb{R}\ \ \mathrm{sign}(u-c)(u + \mathfrak{f}_x(x,c) - g) + \mathfrak{q}(x;u,c)_x \le 0\ \ in\ \mathcal{D}'\big(\mathbb{R}\setminus\{0\}\big), \tag{2.5}$$

and if for every function $c(\cdot)$ of the form (2.4) *it satisfies the global adapted entropy inequalities:*

$$\mathrm{sign}(u-c(x))(u + \mathfrak{f}_x(x,c(x)) - g) + \mathfrak{q}(x;u,c(x))_x \le 0\quad in\ \mathcal{D}'(\mathbb{R}). \tag{2.6}$$

Remark 2.6 *In the homogeneous case (see [7]) one can replace $\mathcal{G}^*$ by $\mathcal{G}$ in* (2.4) *for the evolution problem* (EvPb)*. This weaker assumption leads to a smaller number of global adapted entropy inequalities to be checked. E.g., in the situation where the fluxes $f^{l,r}$ are "bell-shaped", only one global adapted entropy inequality is needed in* (2.3)*, see [17, 4, 5].*

In the present paper, one can replace $\mathcal{G}^*$ by $\mathcal{G}$ in the above definition for the stationary problem (StPb) *but not for the evolution problem. At the present stage, this drawback appears to be the price to pay for the approach which does not rely upon the existence of strong interface traces for solutions of* (EvPb) *(see also [7, Sect. 3.4]).*

Here is the main result of this paper.

Theorem 2.7 (Well-posedness for (EvPb)**)**
Assume $f^{l,r}$ satisfy (H1)–(H3)*. Let $\mathcal{G}$ be a definite maximal L^1D germ. Then for all $u_0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ there exists a unique $\mathcal{G}$-entropy solution of* (EvPb) *with the initial datum u_0. It depends continuously on u_0, namely, if $u,\hat{u}$ are the $\mathcal{G}$-entropy solutions corresponding to $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ data $u_0,\hat{u}_0$, respectively, then $\|u(t,\cdot)-\hat{u}(t,\cdot)\|_{L^1} \le \|u_0-\hat{u}_0\|_{L^1}$ for all $t\in[0,T]$.*

As stated in the introduction, the uniqueness claim is shown in an indirect way, with the help of abstract tools of the nonlinear semigroup theory. The existence can also be obtained in the abstract way, as in [14]. However, here we prefer to justify the existence by constructing solutions with a well-chosen finite volume scheme. Alternatively, in the cases where $\mathcal{G}$ is compatible with some vanishing viscosity approach, the adapted viscosity approximation can be used.

Exploiting the property of finite speed of propagation and a continuation argument for entropy solutions in which we solve auxiliary Dirichlet problems, we can extend the result to general L^∞ data. Namely, we get

Corollary 2.8 *Under the assumptions of Theorem 2.7, the existence and uniqueness of a $\mathcal{G}$-entropy solution still holds if $u_0 \in L^\infty(\mathbb{R})$. If u is the $\mathcal{G}$-entropy solution with $u(t,0) = u_0$, then for all $t \in [0,T]$ the function $u(t,\cdot)$ depends continuously on u_0 in the $L^1_{loc}(\mathbb{R})$ topology.*

In the opposite direction, starting from Theorems 3.7,2.7 we can drop the L^∞ assumption on the data. Indeed, Theorem 3.7 permits to define solutions of the abstract evolution problem (AbEv) for merely L^1 data. In the context of conservation laws of the form $u_t + \operatorname{div}_x f(u) = 0$, such solutions can be characterized intrinsically as its *renormalized solutions* (see [13]). We expect that for general L^1 data, the integral solutions of (AbEv) are renormalized solutions of (EvPb); but this issue is beyond the scope of this paper.

3 The stationary problem (StPb) and the underlying m-accretive operator

Let us define the operator $A_{\mathcal{G}}$ on $L^1(\mathbb{R})$ by its graph:

$$(u,z) \in A_{\mathcal{G}} \ \text{ iff } u \text{ is a } \mathcal{G}\text{-entropy solution of (StPb) with } g = z + u. \tag{3.1}$$

Thus, the domain $D(A_{\mathcal{G}})$ is defined implicitly. Let us show that it consists of trace-regular functions.

Lemma 3.1 (Trace-regularity)
If $u \in L^\infty(\mathbb{R})$ verifies the away-from-the-boundary Kruzhkov entropy inequalities (2.5) *and $f^{l,r}$ verify* (H2),(H3), *then $\gamma^l u := \lim_{x\uparrow 0} u(x)$ and $\gamma^r u := \lim_{x\downarrow 0} u(x)$ exist.*

Proof : Consider, for instance, $u|_{\mathbb{R}^l}$. From entropy inequalities (2.5) it follows that for all $c \in \mathbb{R}$ there exist non-negative Borel measures γ_c^+ on $\mathbb{R}^l = (-\infty, 0)$ such that

$$\operatorname{sign}^+(u-c)(u + \mathfrak{f}_x(x,c) - g) + Q_c(x)_x = -\gamma_c^+ \tag{3.2}$$

where $Q_c(x) := \operatorname{sign}^+(u(x)-c)(f^l(x,u(x)) - f^l(x,c))$. Because $Q_c(x) \in L^\infty(\mathbb{R}^l)$, it is easy to see that the variation of γ_c^+ is finite up to the boundary. Indeed, taking (by approximation) the test function

$$\xi_h(x) = (1 - \min\{1, -x/h\}) \min\{1, (1+x)^+\}$$

in the entropy formulation, we find

$$|\gamma_c^+|([-1,0)) = \lim_{h\to 0} \int_{[-1,0)} \xi_h \, d\gamma_c^+ \int_{-1}^0 |u-g| + \int_{-\infty}^0 |Q_c(x)|\,|(\xi_h)_x|\,dx.$$

The right-hand side is finite, since $\|(\xi_h)_x\|_1 \le 2$ uniformly in $h \in (0,1)$.

Now, let $M = \|u\|_\infty$ and $c_0, \ldots, c_N$ be a partition of $[-M,M]$ such that f' keeps constant sign on each interval (c_{i-1}, c_i), $i = 1, \ldots, N$ (this is possible due to (H2)). For instance, assume that this sign is "$-$" if i is odd and "+" if i is even. Then the variation function (Vf^l) on $[-M,M]$ can be represented as

$$(Vf^l)(x,u) := \int_{-M}^u |(f^l)_u(x,z)|\,dz = \int_{-M}^u \eta'(z)(f^l_u)(x,z)\,dz$$

where $\eta'|_{(c_{i-1},c_i)} = (-1)^i$. Then (Vf^l) is the entropy-flux corresponding to the (non-convex) entropy η with

$$\eta'(z) = sign^+(z-c_0) + 2\sum_{i=1}^{N-1} (-1)^i \operatorname{sign}^+(z-c_i),$$

hence a linear combination of equalities (3.2) yields

$$(Vf^l)(x,u(x))_x = \gamma_{c_0}^+ - 2\sum_{i=1}^{N-1}(-1)^i \gamma_{c_i}^+ - \eta'(u)(u - g + \mathfrak{f}_x(x,c)) \ \text{ in } \mathcal{D}'(-\infty,0).$$

From the facts that $(u-g) + \mathfrak{f}_x(u,x) \in L^1(\mathbb{R}^l) + L^\infty(\mathbb{R}^l)$ and that γ_{c_i} are finite up to the boundary, it follows that $(Vf^l)(x,u(x)) \in C((-\infty,0])$. Now, notice that the map $W(\cdot) := (Vf^l)(0,\cdot)$ is non-decreasing, by construction;

moreover, due to the assumption (H3) the map W is strictly increasing (and furthermore, we can assume that it is bijective, upon modifying the definition of W outside $[-M, M]$). Therefore the map $x \mapsto W^{-1} \circ (Vf^l)(x, u(x))$ is continuous on $(-\infty, 0]$. Hence its limit at zero exists; let us denote it by $\gamma^l u$.

It remains to notice that $\gamma^l u = \lim_{x \uparrow 0} u(x)$. Indeed, because f^l is continuous in $(x, u) \in \mathbb{R}^l \times \mathbb{R}$, this is also the case of Vf^l. Moreover, $W^{-1} \circ (Vf^l)(0, \cdot)$ is the identity map. Hence

$$|u(x) - W^{-1} \circ (Vf^l)(x, u(x))| = |W^{-1} \circ (Vf^l)(0, u(x)) - W^{-1} \circ (Vf^l)(x, u(x))|$$

vanishes as $x \to 0$ (notice that $u(x)$ stays in a compact set on which W^{-1} is uniformly continuous). This concludes the proof. $\square$

Now, we can reformulate Definition 2.5 as follows.

Lemma 3.2 (Interface coupling for (StPb)**)**
Assume $(H2), (H3)$. *A function* $u \in L^\infty(\mathbb{R})$ *is a* $\mathcal{G}$*-entropy solution of* (StPb) *if and only if it satisfies* (2.5) *and, in addition,* $(\gamma^l u, \gamma^r u) \in \mathcal{G}^*$.

Note that by Lemma 3.1 the existence of $\gamma^{l,r} u$ is automatic in the above statement.

PROOF : Let us prove that an entropy solution of (StPb) verifies $(\gamma^l u, \gamma^r u) \in \mathcal{G}^*$. It is enough to take $\xi_h = 1 - \min\{|x|/h, 1\}$ as test function in (2.6) and let $h \to 0$; one finds

$$\forall (c^l, c^r) \in \mathcal{G}^* \;\; q^l(0, \gamma^l u, c^l) - q^r(0, \gamma^r u, c^r) \geq 0. \tag{3.3}$$

Because $\mathcal{G}^*$ is a maximal L^1D germ associated with the fluxes $f^{l,r}(0, \cdot)$, the claim follows.

Reciprocally, by the definition of an L^1D germ, the property $(\gamma^l u, \gamma^r u) \in \mathcal{G}^*$ implies (3.3). It remains to take $(1 - \xi_h)\xi$ as a test function in (2.5), where $\xi \in \mathcal{D}(\mathbb{R})$. One deduces (2.6). $\square$

Now, let us study the operator $A_\mathcal{G}$. We refer to [12, 14, 16] for the definitions.

Proposition 3.3 (Accretivity)
Let $\mathcal{G}$ *be a definite* L^1D *germ. Assume* $f^{l,r}$ *satisfy* $(H2), (H3)$. *Then the operator* $A_\mathcal{G}$ *is accretive on* $L^1(\mathbb{R})$.

PROOF : One has to prove that for all $(u, z), (\hat{u}, \hat{z}) \in A_\mathcal{G}$ there holds

$$\forall \lambda > 0 \;\; \|u - \hat{u}\|_{L^1} \leq \|(u + \lambda z) - (\hat{u} + \lambda \hat{z})\|_{L^1}. \tag{3.4}$$

It is easily seen that u,$\hat{u}$ are $\mathcal{G}$-entropy solutions of the stationary problem (StPb) with the flux $\lambda\mathfrak{f}$ in the place of $\mathfrak{f}$ and with the source terms $h = u + \lambda z$, $\hat{h} = \hat{u} + \lambda \hat{z}$, respectively. For instance, the entropy inequality (2.5) with $g = u + z$ can be rewritten as

$$\operatorname{sign}(u - c)\big(u - (u + \lambda z - \lambda \mathfrak{f}_x(x, c))\big) + \lambda \mathfrak{q}(x, u, c)_x \leq 0. \tag{3.5}$$

Based on (3.5) and its analogue written for $\hat{u}$, we can use the Kruzhkov doubling of variables to deduce the so-called *Kato inequality*:

$$|u - \hat{u}| + \lambda \mathfrak{q}(x, u, \hat{u})_x \leq |h - \hat{h}| \;\text{ in } \mathcal{D}'(\mathbb{R} \setminus \{0\}). \tag{3.6}$$

The argument we use to derive this inequality is essentially based on the fundamental work of Kruzhkov [22], but it is not entirely classical. Indeed, notice that we have the dependency of $\mathfrak{f}$ on x but we are able to drop "$\mathfrak{f}_x(x, c)$" term that appears in [22]. Roughly speaking, we justify that a Kruzhkov entropy solution (even a local one!) is a vanishing viscosity limit; and we observe that the solution operator for $u + f(x, u)_x - \varepsilon u_{xx} = h$ leads to a Kato inequality which limit, as $\varepsilon \to 0$, brings (3.6). The details of justification of (3.6) are postponed to the Appendix (see in particular Remark 5.3).

Then it remains to take the test function $\xi_s(x) = exp(-s|x|) \min\{1, |x|/s\}$ in (3.6); this can be done by approximation. Taking into account the fact that $|q(x, u, \hat{u})| \leq L|u - \hat{u}|$ where L is a uniform in x Lipschitz constant of $\mathfrak{f}(x, \cdot)$ (here we use (H2)), at the limit $s \to 0^+$ we infer

$$\|u - \hat{u}\|_{L^1} \leq \|h - \hat{h}\|_{L^1} - \big(q^l(0, \gamma^l u, \gamma^l \hat{u}) - q^r(0, \gamma^r u, \gamma^r \hat{u})\big) \leq \|h - \hat{h}\|_{L^1};$$

the latter inequality follows by Lemma 3.2 and the L^1-dissipativity of $\mathcal{G}^*$. In view of the definition of $h, \hat{h}$, this proves (3.4). $\square$

Proposition 3.4 (*m-accretivity of the closure of $A_{\mathcal{G}}$*)
Let $\mathcal{G}$ be a complete maximal L^1D germ. Assume $f^{l,r}$ satisfy (H1)–(H3).
(i) We have $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R}) \subset Im(I + \lambda A_{\mathcal{G}})$, for all $\lambda > 0$.
(ii) The domain $D(A_{\mathcal{G}})$ is dense in $L^1(\mathbb{R})$.

PROOF : For the proof of (i), we construct solutions of $u + \lambda A_{\mathcal{G}} u = g$ for $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ using a monotone two-point finite volume scheme, in the vein of [7, Th. 6.4]. See Remark 3.5 for an alternative construction. For the proof of (ii), we denote by u^λ the solution of the problem treated in the first part; letting $\lambda \to 0$, we will prove the convergence of u^λ to g for an L^1-dense set of source terms g. Now, let us give the details.

Let us approximate problem $u + \mathfrak{f}(x,u)_x = g$ (indeed, it is enough to consider $\lambda = 1$) by piecewise constant functions $u_h := \sum_{n=-\infty}^{+\infty} u_n 1\!\!1_{((n-1)h,nh)}$ using a finite volume scheme. To this end, discretize $x \mapsto \mathfrak{f}(x,\cdot)$ by

$$f_n(z) = f^l(nh,z) \text{ if } n < 0 \quad \text{and} \quad f_n(z) = f^r(nh,z) \text{ if } n > 0.$$

For every $n \neq 0$, we take a monotone two-point flux F_n (see, e.g., [21]) consistent with f_n. Since $\mathcal{G}$ is a complete germ, for $i = 0$ we can take the Godunov flux F_0 associated with the Riemann solver for the auxiliary discontinuous-flux problem (2.1) associated with the fluxes $f^{l,r}(0,\cdot)$ (cf. [7, Sect. 6.3]). Now, the finite volume scheme to be solved writes

$$\forall n \in \mathbb{Z} \;\; u_n + h\big(F_n(u_{n+1},u_n) - F_{n-1}(u_n,u_{n-1})\big) = g_n \tag{3.7}$$

where $g_h := \sum_{n=-\infty}^{+\infty} g_n 1\!\!1_{((n-1)h,nh)}$ is an approximation of g in $L^1(\mathbb{R})$ such that $\|g_h\|_{L^1} \leq \|g\|_{L^1}$, $\|g_h\|_{L^\infty} \leq \|g\|_{L^\infty}$ and $\|g_h\|_{BV} \leq \|g\|_{BV}$.

Due to (H1),(H2) we can choose F_n Lipschitz continuous in both variables, uniformly in n. Therefore, for h small enough, the scheme can be rewritten under the form

$$\forall n \in \mathbb{Z} \;\; H_n(u_{n-1},u_n,u_{n+1}) = g_n \;\text{ with } H_n \text{ monotone in each variable.}$$

From this property and assumption (H1) we get the uniform L^∞ *a priori* bound $\min\{0,m\} \leq u_h \leq \max\{1,M\}$ where m, M are such that $m \leq g_h \leq M$ a.e. on $\mathbb{R}$.

Existence of a solution to the scheme can be inferred from the topological degree theorem as follows. One first truncates the system at ranks $\pm N$, setting $u_{-N} = 0 = u_N$ and considering only the equations for $|n| < N$ with F_n, g_n substituted by θF_n, θg_n, respectively, where $\theta \in [0,1]$. For $\theta = 0$ the problem has the trivial zero solution. The a priori L^∞ estimate still holds for the truncated problem, and the topological degree theorem ensures existence of a solution $U^N \in \mathbb{R}^{2N-1}$ (for $\theta = 1$) to the finite-dimensional system. We consider U^N as an element of $\mathbb{R}^{\mathbb{Z}}$, setting to zero the components with $|n| \geq N$. Then the compactness (component per component) and the diagonal extraction are used to obtain an accumulation point $U := \lim_{N_k \to \infty} U^{N_k}$ in the topology of component-wise convergence in $\mathbb{R}^{\mathbb{Z}}$. Then by passage to the limit in the truncated problem, it is easily seen that $U = (u_n)_{n \in \mathbb{Z}}$ solves problem (3.7).

Now we have to prove that, first, there exists a convergent subsequence $(u_h)_h$ (not labelled); and second, that $u := \lim_{h \downarrow 0} u_h$ is a $\mathcal{G}$-entropy solution of (StPb).

Let us assess the $BV_{loc}(\mathbb{R} \setminus \{0\})$ compactness of $(u_h)_h$. We can restrict our attention to $h \in \{2^{-j} \,|\, j \in \mathbb{N}\}$. Let us normalize u_h so that it is left-continuous for $x < 0$ and right-continuous for $x > 0$. Using the diagonal extraction argument we can ensure that $u^h(\pm 2^{-\ell})$ converge to some limits $u^\pm_\ell$ as $h \to 0$, for all $\ell \in \mathbb{N}$. Similarly, we can assume that $u^h(\pm 2^\ell \mp 0) \to U^\pm_\ell$ as $h \to 0$. Then we can consider that u^h approximate the Dirichlet boundary-value problems in $(-2^\ell, -2^{-\ell})$ (with the boundary values U^-_ℓ and u^-_ℓ at the extremities) and in $(2^{-\ell}, 2^\ell)$ (with the boundary values u^+_ℓ and U^+_ℓ). By standard arguments (see in particular [21] and [17, 7]) using the monotonicity of H_n and the fact that $\sup_{a,b,c \in [m,M]} |H_n(a,b,c) - H_{n-1}(a,b,c)| \leq const\, h$ (this comes from (H1),(H2)) we deduce a uniform BV bound on $(u^h)_h$ in $\{x \in \mathbb{R} \,|\, 2^{-\ell} < |x| < 2^\ell\}$. Another application of the diagonal extraction argument proves the BV_{loc} compactness in $\mathbb{R} \setminus \{0\}$.

It remains to pass to the limit in the scheme, as $h \to 0$. Thanks to the local variation bound, this is a standard issue (see [21] and the arguments of [7] for the discontinuous-flux context). One first gets approximate entropy inequalities and approximate adapted entropy inequalities for u^h; here, it is important that we use the Godunov flux at the interface. Then one sends h to zero using the L^1_{loc} compactness of $(u_h)_h$. In particular, consistency of the numerical fluxes and the continuity of $f^{l,r}$ in x permit to pass to the limit in the nonlinear terms. This concludes the proof of (i).

Now, we turn to the proof of (ii). Indeed, let g be a compactly supported, piecewise constant function. We will use λ-dependent test functions ψ_λ on each interval where g is constant. Namely, let $g = c_i$ on a finite or semi-infinite interval (a_{i-1}, a_i); without loss of generality we may assume that $0 \notin (a_{i-1}, a_i)$. From the Kruzhkov entropy inequalities for u^λ which is a $\mathcal{G}$-entropy solution of $u + \lambda\mathfrak{f}(x,u)_x = g$, we have

$$\operatorname{sign}(u^\lambda - c_i)\big(u^\lambda - c_i + \lambda\mathfrak{f}_x(x, u^\lambda, c_i)\big) + \lambda\mathfrak{f}(x, u^\lambda)_x \le 0 \ \text{ in } \mathcal{D}'((a_{i-1}, a_i)).$$

Taking test functions ψ_λ in this inequality such that $\psi_\lambda \to 1\!\!1_{(a_{i-1},a_i)}$ with $\|\psi'_\lambda\|_\infty \le \lambda^{-1/2}$, we find

$$\lim_{\lambda\downarrow 0} \int_{a_{i-1}}^{a_i} |u^\lambda - g| = \lim_{\lambda\downarrow 0} \int_{a_{i-1}}^{a_i} |u^\lambda - c_i| \le 0.$$

Summing in i, we deduce that $u^\lambda \to g$ in $L^1(\mathbb{R})$ as $\lambda \to 0$. This ends the proof. □

Remark 3.5 *Notice that in many cases, existence of a $\mathcal{G}$-entropy solution can be shown using an* adapted vanishing viscosity *approximation.*

For instance, in the case of bell-shaped fluxes, one looks at the definite germs of the form $\mathcal{G}_{(A,B)} = \{(A,B)\}$ where (A,B) are the so-called connections *(see [2, 17, 5]). For each of these germs, there exists a choice of* adapted viscosity approximations *that take the form*

$$u^\varepsilon + \mathfrak{f}(x, u^\varepsilon)_x = g + \varepsilon(\mathfrak{a}(x, u^\varepsilon))_{xx},$$

and for which $u = A1\!\!1_{x<0} + B1\!\!1_{x>0}$ is an obvious solution with $g = u + f^l_x(x,A)1\!\!1_{x<0} + f^r_x(x,B)1\!\!1_{x>0}$, for every $\varepsilon > 0$. As in [7, Th. 6.3], one deduces the convergence of u^ε to a $\mathcal{G}$-entropy solution u of (StPb)*. Moreover, one can use viscosity approximations having the physical meaning of* vanishing capillarity*, see [5].*

Recall the definition of an integral solution for an evolution equation governed by an accretive operator on L^1.

Definition 3.6 (Integral solution)
A function $u \in C([0,T], L^1(\mathbb{R}))$ is an integral solution of $\frac{d}{dt}u + Au \ni h$ with A defined on $L^1(\mathbb{R})$ if $u(0) = u_0$ and (1.5) *holds in $\mathcal{D}'(0,T)$, with the notation $\big[u, f\big]_{L^1} := \int \operatorname{sign} u\, f \,+ \int |f|\, 1\!\!1_{u=0}$.*

Now we can apply the key result of the nonlinear semigroup theory.

Theorem 3.7 (Uniqueness of an integral solution)
Assume (H1)–(H3)*. For all $u_0 \in L^1$ there exists one and only one integral solution to the problem $\frac{d}{dt}u + \overline{A_{\mathcal{G}}}u \ni 0$ with the initial datum u_0. If $\hat u$ is the integral solution corresponding to $\hat u_0$, then $\|u(t) - \hat u(t)\|_{L^1} \le \|u_0 - \hat u_0\|_{L^1}$.*

Proof : It is enough to apply [14, Th. 6.6] to the closure of $A_{\mathcal{G}}$. Indeed, according to Propositions 3.3,3.4, $\overline{A_{\mathcal{G}}}$ is a densely defined m-accretive operator. Therefore there exists a mild solution to the abstract evolution problem governed by $\overline{A_{\mathcal{G}}}$; hence the mild solution is the unique integral solution of this problem. □

4 $\mathcal{G}$-entropy solutions of the evolution problem

In this section, the main issue is the uniqueness of a solution to (EvPb) in the sense of Definition 2.3. We first derive an equivalent form of this definition (note the difference with the stationary case: we do not ensure nor exploit the trace-regularity of u solution of (EvPb)).

Lemma 4.1 (Interface coupling for (EvPb)**)**
Assume (H2),(H3)*. A function $u \in L^\infty(\mathbb{R})$ is a $\mathcal{G}$-entropy solution of* (EvPb) *iff it satisfies* (2.2) *and, in addition,*

$$\forall (c^l, c^r) \in \mathcal{G}^* \ \ (\gamma^l_w q^l(\cdot, u(\cdot), c^l))(t) \ge (\gamma^r_w q^r(\cdot, u(\cdot), c^r))(t) \ \text{ for a.e. } t \in (0,T). \tag{4.1}$$

Here $\gamma^{l,r}_w q^{l,r}(u, c^{l,r})$ denote the weak interface traces *of the respective fluxes.*

Note that the existence of $\gamma^{l,r}_w q^{l,r}(\cdot, u(\cdot), c^{l,r})$ comes from the Kruzhkov entropy inequalities (2.2), the Schwartz lemma on non-negative distributions and the general result of [20]. At this point, it should be stressed that the left-hand side of (2.2) is a non-positive Radon measure that is, in addition, finite up to the interface $\{x = 0\}$ (cf. the corresponding argument of the proof of Lemma 3.1).

Proof : As in the proof of Lemma 3.2, we use $\xi_h = 1 - \min\{|x|/h, 1\}$. Taking $\xi_h(x)\theta(t)$ (with $\theta \in \mathcal{D}(0,T)$, $\theta \geq 0$) as test function in (2.3), using the existence of weak traces $\gamma_w^{l,r} q^{l,r}(u, c^{l,r})$ we find the $\mathcal{D}'$ formulation of (4.1). Since θ is arbitrary, we get (4.1) by localization at every point of $(0,T)$ that is a Lebesgue point of the weak trace functions $t \mapsto (\gamma^{l,r} q^{l,r}(u, c^{l,r}))(t)$. Reciprocally, in the way similar to Lemma 3.2, it can also be shown that (4.1) and (2.2) imply (2.3). □

As it was the case for Lemma 3.2, Lemma 4.1 provides an equivalent definition of $\mathcal{G}$-entropy solution.

Now, note the following elementary property.

Lemma 4.2 *Let u, $\hat{u}$ be two bounded functions for which we assume that*

$$\text{the weak interface traces} \quad \gamma_w^{l,r} q^{l,r}(\cdot, u(\cdot), \hat{u}(\cdot)) \quad \text{exist.}$$

If $\hat{u}$ is a trace-regular function (i.e., there exist strong interface traces $(\gamma^{l,r}\hat{u})(t)$ for a.e. $t \in (0,T)$), then

$$\gamma_w^{l,r} q^{l,r}(\cdot, u(\cdot), \hat{u}(\cdot)) = \gamma_w^{l,r} q^{l,r}(\cdot, u(\cdot), c^{l,r}) \text{ with } c^{l,r}(t) = (\gamma^{l,r}\hat{u})(t), \text{ for a.e. } t. \tag{4.2}$$

Proof : Property (4.2) stems for the definition of a weak trace in the L^∞ sense (actually, this is a weak-* sense) and the fact that due to the continuity of $f^{l,r}$ and the existence of strong traces, one has for instance

$$\operatorname*{ess\,lim}_{x \uparrow 0} |q^l(x, u(t,x), \hat{u}(t,x)) - q^l(x, u(t,x), c^l(t))| = 0 \text{ for a.e. } t$$

while $q^l(x, u, \hat{u})$ remains uniformly bounded. □

We are now in a position to deliver the key observation of our method:

Proposition 4.3 *Assume* (H2). *Let u be a $\mathcal{G}$-entropy solution u of* (EvPb) *with $u(0,\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the map $t \mapsto u(t,\cdot)$ is an integral solution of the associated abstract evolution problem governed by the operator $\overline{A_\mathcal{G}}$ (with $h = 0$ and the initial datum $u(0,\cdot)$, cf. Remark 2.4).*

Proof : By density argument and the upper semi-continuity in $L^1(\mathbb{R})$ of the bracket $[\cdot,\cdot]_{L^1}$, it is enough to prove (1.5) (i.e., we can consider only $(u,z) \in A_\mathcal{G}$ in the place of $(u,z) \in \overline{A_\mathcal{G}}$). Recall that we have $h = 0$.

By the definition (3.1) of $A_\mathcal{G}$, we take $\hat{u}$, a $\mathcal{G}$-entropy solution of the stationary problem (StPb). Then we "compare" u and $\hat{u}$ using the Kruzhkov doubling of variables: more precisely, we use it *away from the interface*. Using the version of the Kruzhkov argument presented in Appendix, we deduce the local (in $\mathbb{R} \setminus \{0\}$) *Kato inequality*

$$|u - \hat{u}|_t + \mathfrak{q}(x, u, \hat{u})_x \leq \operatorname{sign}(u - \hat{u})(u - g) + 1\!\!1_{[u = \hat{u}]}|g - \hat{g}| = [u(t) - \hat{u}, u - g]_{L^1} \text{ in } \mathcal{D}'(R \setminus \{0\}). \tag{4.3}$$

Letting the test function in (4.3) converge to 1 in the same way as in the proof of Proposition 3.3, we make appear the weak interface traces:

$$\begin{aligned} &\frac{d}{dt}\|u(t) - \hat{u}\|_{L^1} \leq [u(t) - \hat{u}, u - g]_{L^1} - \Delta, \\ &\text{with } \Delta := \int_0^T \left((\gamma_w^l q^l(\cdot, u(\cdot), \hat{u}(\cdot)))(t) - (\gamma_w^r q^r(\cdot, u(\cdot), \hat{u}(\cdot)))(t)\right) dt. \end{aligned} \tag{4.4}$$

It remains to combine Lemma 4.2 (note that $\hat{u}$ is trace-regular, by Lemma 3.1), Lemma 3.2 and Lemma 4.1. One finds that the term Δ in (4.4) is non-negative, which leads to inequalities (1.5). Indeed, we have $c^{l,r} = \gamma^{l,r}\hat{u}$ that fulfill $(c^l, c^r) \in \mathcal{G}^*$; then Δ can be re-written using (4.2); eventually, (4.1) guarantees that the integrand in Δ is non-negative. This ends the proof. □

Proof of Theorem 2.7: The uniqueness claim and the L^1-contraction property are straightforward from Proposition 4.3 and Theorem 3.7. In order to conclude the proof of Theorem 2.7, it remains to ensure the existence of an entropy solution. We refer to the existence arguments used for the stationary problem (see Proposition 3.4(i) and Remark 3.5). For the evolution problem, analogous approximation arguments apply: either approximation by a finite volume scheme or, in the case of bell-shaped fluxes, the use of adapted viscosity approximations. One should pay attention to heterogeneity, as in the proof of Proposition 3.4(i) and in Remark 3.5. The delicate point is the BV_{loc} estimate which proof is more involved than the arguments used to justify Proposition 3.4(i); one has to argue in the same way as in [17, 7]. □

5 On the Dirichlet problem for one-dimensional conservation law

5.1 Application of the semigroup method to the Dirichlet problem

The fundamental reference for the Dirichlet problem

$$\begin{cases} u_t + f(x,u)_x = 0 \ \text{ in } (0,T)\times(0,+\infty) \\ u|_{x=0} = u^D \\ u|_{t=0} = u_0 \end{cases} \tag{5.1}$$

is the Bardos, LeRoux and Nédélec paper [11]. The setting of [11] is the $L^\infty(0,T;BV((0,+\infty))$ space, thus $u_0 \in BV(0,+\infty)$ and $u^D \in BV(0,T)$; moreover, f should be BV in x. These restrictions are due to the fact that the formulation of [11] uses strong boundary trace γu of u on $\{x=0\}$. More recently, Vasseur [29] (see also [27] for the most general argument) proved existence of such traces for the spatially homogeneous case and thus dropped the BV assumptions of [11]. Notice that the result of [11] is used[4] in our proof of Theorem 2.7 through the justification of Lemma 5.5 in the Appendix; thus we have kept the BV assumption on f.

For the non-homogeneous case $f = f(x,u)$, with the method as in the present paper we can treat the particular case where u^D is a constant in t function (this restriction is inherent to the semigroup approach). To do so, we can exploit the notion of solution for (5.1) based upon the up-to-the-boundary entropy inequalities introduced in [8]. The arguments of the well-posedness proof are almost identical to those developed for problem (EvPb); the use of a germ is replaced by the use of some maximal monotone graph which encodes a boundary dissipation property analogous to (1.3).

Yet let us stress that the method of weak boundary trace formulation (Otto, [25, 24]; see also the slightly different definition in [30]) gives the general well-posedness result for the Dirichlet problem (5.1); indeed, the case of non-homogeneous flux function $f = f(t,x,u)$ has been treated in the work of Vallet [28]. In an opposite direction, we refer to [8] for a throughful treatment of conservation laws with different nonlinear boundary conditions, in the case of a homogeneous flux $f = f(u)$ and in the strong trace setting. Our argument can be used in the setting of [8] with $f = f(x,u)$, for various boundary conditions.

5.2 Continuation of local entropy solutions and justification of Corollary 2.8

Let us justify the extension to L^∞ data of the results obtained for $L^1 \cap L^\infty$ ones. To this end, we exploit the Dirichlet problem (in its strong-trace formulation) for conservation laws with (x,u)-continuous flux.

Proof of Corollary 2.8 (sketched): The existence arguments for Theorem 2.7 do not require the L^1 assumption on the data, hence there is nothing to be generalized at this point.

In order to deduce the uniqueness and the continuous dependence on the data for (EvPb) with L^∞ data, we use the property of finite speed of propagation. Indeed, let u be a $\mathcal{G}$-entropy solution of (EvPb) with some L^∞ datum. Firstly, applying the result of [22] (for conservation laws in Ω^l and Ω^r) we readily see that the solution is uniquely defined by the datum outside the triangle $\mathcal{T} := \{(t,x)\,|\, t \in (0,T],\, |x| \le Lt\}$ where $L = L_0 + 1$ and L_0 is the uniform in x Lipschitz constant of the flux $\mathfrak{f}(x,\cdot)$. To prove uniqueness of the solution in $\mathcal{T}$, we construct another $\mathcal{G}$-entropy solution $\tilde u$ that coincides with u in $\mathcal{T}$ but which corresponds to an $L^1 \cap L^\infty$ initial datum. Let us give the idea of the construction and sketch the details, that require some careful analysis of the Dirichlet problem for non-homogeneous conservation laws with a "space-like" boundary[5].

For $h > 2LT$, consider the segments $S_h^\pm := \{x = \pm(h - Lt),\, t \in [0,T]\}$. A.e. $h > 0$ is a Lebesgue point of the maps $h \mapsto u|_{S_h^\pm}$ with values in L^1. Thus, we can pick $h_0 > 2LT$ such that strong traces of u on both $S_{h_0}^+$ and $S_{h_0}^-$ exist. Then we set $\tilde u \equiv u$ for $t \in [0,T]$ and $|x| \le h_0 - Lt$ (note that this domain contains $\mathcal{T}$, by the choice of h_0). We extend $\tilde u$ to the remaining part of the strip $[0,T] \times \mathbb{R}$ by solving two Cauchy-Dirichlet problems with fluxes $f^l(x,\cdot)$ (for $x < 0$) and $f^r(x,\cdot)$ (for $x > 0$). For instance, in the domain where $x < -(h_0 + Lt)$ we take the flux $f^l(x,\cdot)$, use the zero initial datum and the boundary datum which is the strong trace γu on $S_{h_0}^-$. To construct the solution in the domain with slanted boundary, it is enough to change the variables. Setting $y = x - Lt + h_0$, in variables (t,y) we obtain a new conservation law in the domain $\Theta = (0,T) \times (-\infty,0)$,

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[4] to be specific, the Bardos-LeRoux-Nédélec formulation with a strong boundary trace (cf. [29]) is used not in Ω but in specially selected subdomains of Ω, so that the existence of strong boundary traces comes "for gratis"

[5] Consider a conservation law of the form $\operatorname{div}_{(t,x)} \phi(t,x,u) = h(t,x)$ set up in a space-time domain Q. We say that the boundary ∂Q is *space-like* if the map $u \mapsto \phi(t,x,u)\cdot n(t,x)$ is strictly decreasing for all point (t,x) of the boundary. In this case, the local change of variables $w(t,x) := \phi(t,x,u)\cdot n(t,x)$ (the field of exterior unit normal vectors $n(\cdot)$ on ∂Q should be lifted in a neighbourhood of ∂Q) reduces the situation to a standard conservation law with the time direction given by the vector field $n(\cdot)$.

moreover, its characteristics are outgoing on the boundary (this is due to the choice of L and to the change of variable we make). For instance, the result of [28] ensures that there exists a solution to such Cauchy-Dirichlet problem in the domain Θ. Moreover, because the boundary is space-like it can be shown that the solution assumes, in the strong sense, the Dirichlet datum that was prescribed on the boundary[6]. Consider the domain Ω^l; now $\tilde{u}|_{\Omega^l}$ is the juxtaposition of two Kruzhkov entropy solutions on the two sides from the segment $S^-_{h_0}$. It is a Kruzhkov entropy solution, due to the continuity of $\tilde{u}$ that we enforced across the segment $S^-_{h_0}$. In the same way, we see that $\tilde{u}$ is a Kruzhkov entropy solution in the domain Ω^r. Moreover, the trace property (4.1) for u is inherited by $\tilde{u}$. Thus, using the characterization of Lemma 4.1 we see that $\tilde{u}$ is indeed a $\mathcal{G}$-entropy solution of (EvPb) corresponding to the truncated initial datum $\tilde{u}_0 = u_0 1\!\!1_{[-h_0,h_0]}$. Further, by assumption (H1) it is easy to deduce that, whatever be the L^1D germ $\mathcal{G}$, the couples (r,r) with $r \notin (0,1)$ belong to $\mathcal{G}$. Then from the entropy formulation one readily gets the $L^\infty(0,T;L^1(\mathbb{R}))$ bound on $\tilde{u}$.

Now we are in a position to apply the result of Theorem 2.7. Given two solutions u and $\hat{u}$ with the same initial datum, we obtain $\tilde{u}$, $\tilde{\hat{u}}$ to which the result of the theorem applies (notice that a common value of h_0 can be taken while constructing $\tilde{u}$ and $\tilde{\hat{u}}$). This ensures that u and $\hat{u}$ coincide in between the segments $S^-_{h_0}$ and $S^+_{h_0}$, thus they coincide in the triangle $\mathcal{T}$. This ends the proof of uniqueness. Coming back to the same arguments but using different initial data, we readily deduce an L^1_{loc} estimate of $u(t,\cdot) - \hat{u}(t,\cdot)$ in terms of the L^1_{loc} distance between u_0 and $\hat{u}_0$. □

Appendix

Throughout the Appendix, we assume that

$$f \text{ is a Lipschitz continuous function of } (t,x,u) \in (0,T) \times \Omega \times \mathbb{R}, \tag{HA}$$

where Ω is an open domain of $\mathbb{R}^N$. Our objective is to prove the following "sharp Kato inequality":

Theorem 5.1 *Assume* (HA). *Let u be a Kruzhkov entropy solution of a conservation law*

$$u_t + \mathrm{div}_x\, f(t,x,u) = g(t,x) \tag{5.2}$$

in $(0,T) \times \Omega$. Let $\hat{u}$ be another Kruzhkov entropy solution corresponding to a source term $\hat{g}$. Then one has in $\mathcal{D}'((0,T) \times \Omega)$ the inequality

$$|u - \hat{u}|_t + \mathrm{div}_x\, \mathrm{sign}(u - \hat{u})\big(f(t,x,u) - f(t,x,\hat{u})\big) \le \mathrm{sign}(u - \hat{u})(g - \hat{g}) + 1\!\!1_{[u=\hat{u}]}|g - \hat{g}|. \tag{5.3}$$

Remark 5.2 *Notice that the "rough Kato inequality" with the additional term Const $|u - \hat{u}|$ in the right-hand side of* (5.3) *can be deduced directly from the doubling of variables approach of Kruzhkov [22]. This additional term originates from a bound on $\big|(\mathrm{div}_x f)(t,x,u) - (\mathrm{div}_x f)(t,x,\hat{u})\big|$; although this latter term is absent from the formal computation, it appears in the proof whenever the regularity of u is not sufficient to write*

$$\mathrm{div}_x\, \mathrm{sign}(u-k)\big(f(x,u) - f(x,k)\big) = \mathrm{sign}(u-k) f_u(x,u) \cdot \nabla u + \mathrm{sign}(u-k)\big((\mathrm{div}_x f)(x,u) - (\mathrm{div}_x f)(x,k)\big).$$

Therefore, we argue at the level of the more regular vanishing viscosity approximations, and then observe that locally, every entropy solution of (5.2) *can be seen as a vanishing viscosity limit.*

Remark 5.3 *Notice that, considering solutions of the stationary problem $u + \mathrm{div}_x f(x,u) = g$ as time-independent solutions of the corresponding conservation law with the source term $h = g - u$, one deduces* (3.6) *from* (5.3).

The proof of Theorem 5.1 is a straightforward combination of the two following lemmas.

Lemma 5.4 *Assume* (HA). *Assume that $u \in L^\infty((0,T) \times \Omega)$ is the L^1_{loc} limit, as $\varepsilon \to 0$, of functions u^ε that are solutions (in the variational sense: namely, $u \in V := L^2(0,T;H^1_{loc}(\Omega))$ with the equation satisfied in the dual space of V) of the viscosity approximated equation* (5.2):

$$u^\varepsilon_t + \mathrm{div}_x\, f(t,x,u^\varepsilon) - \varepsilon \Delta u^\varepsilon = g(t,x). \tag{5.4}$$

Similarly, assume $\hat{u} \in L^\infty((0,T) \times \Omega)$ is the L^1_{loc} limit of functions $\hat{u}^\varepsilon$ that are the viscosity approximations of the corresponding equation with the source term $\hat{g}$. Then (5.3) *holds in $\mathcal{D}'((0,T) \times \Omega)$.*

[6] To justify this claim, the arguments are the same as for the time-continuity of entropy solutions. Indeed, we have ensured that the normal component of the flux is a strictly increasing function: this makes the normal direction to the boundary *time-like*. Let us stress that the existence of strong trace for this case is considerably simpler to justify than for the general one: as a matter of fact, it follows from a local application of entropy inequalities. We refer to [18] and to [4, Lemma A4] for the arguments that can be used in this context.

Lemma 5.5 *Assume* (HA). *Let u be a Kruzhkov entropy solution of a conservation law* (5.2) *in* $(0,T)\times\Omega$. *Then there exists a sequence* $(\omega_n)_n$ *of open subdomains of* Ω *such that* $\Omega=\cup_{n=1}^\infty \omega_n$ *and in each domain* $(0,T)\times\omega_n$*, the function u is the a.e. limit, as* $\varepsilon\to 0$*, of some solutions* u_n^ε *of equations* (5.4) *in the domain* $(0,T)\times\omega_n$.

PROOF OF LEMMA 5.4: The argument is a classical one. One takes $H_\alpha : z\mapsto \int_0^z \frac{1}{\alpha}1\!\!1_{[-\alpha,\alpha]})(s)\,ds$ (this is a Lipschitz approximation of the sign function). Set $I_\alpha : z\mapsto \int_0^z H_\alpha(s)\,ds$; we have $I_\alpha(\cdot)\to|\cdot|$ uniformly on $\mathbb{R}$.

Fix $\xi\in\mathcal{D}((0,T)\times\Omega)$. Take the difference of equations (5.4) written for u^ε and $\hat u^\varepsilon$ and take the test function $H_\alpha(u^\varepsilon-\hat u^\varepsilon)\xi\in L^2(0,T;H^1(\Omega))$ in the corresponding variational formulation. We get

$$\int_0^T\!\!\int_\Omega \big\{-I_\alpha(u^\varepsilon-\hat u^\varepsilon)\,\xi_t - H_\alpha(u^\varepsilon-\hat u^\varepsilon)\big(f(x,u^\varepsilon)-f(x,\hat u^\varepsilon)-\varepsilon(\nabla u^\varepsilon-\nabla\hat u^\varepsilon)\big)\cdot\nabla\xi\big\}$$
$$\le \int_0^T\!\!\int_\Omega H_\alpha(u^\varepsilon-\hat u^\varepsilon)(g-\hat g)\,\xi + \frac{1}{\alpha}\iint_{[0<|u^\varepsilon-\hat u^\varepsilon|<\alpha]}(f(x,u^\varepsilon)-f(x,\hat u^\varepsilon))\cdot\nabla(u^\varepsilon-\hat u^\varepsilon)\xi.$$

Here, we have used two chain rules (see in particular [19]) and the fact that for a.e. t, the gradient of the $H^1(\Omega)$ function $(u^\varepsilon-\hat u^\varepsilon)(t,\cdot)$ is zero a.e. on the set where $u^\varepsilon(t,\cdot)-\hat u^\varepsilon(t,\cdot)=const$. Due to the Lipschitz assumption (HA) the last term of the above inequality vanishes, as $\alpha\to 0$. Indeed, it is bounded by the integral of the L^1 function $Const|\nabla u^\varepsilon-\nabla\hat u^\varepsilon|\xi$ over the set $[0<|u^\varepsilon-\hat u^\varepsilon|<\alpha]:=\big\{(t,x)\,|\,0<|u^\varepsilon(t,x)-\hat u^\varepsilon(t,x)|<\alpha\big\}$ which measure vanishes as $\alpha\to0$. Thus letting $\alpha\to0$ then $\varepsilon\to0$, we deduce (5.3) in $\mathcal{D}'((0,T)\times\Omega)$. $\square$

PROOF OF LEMMA 5.5: We will select ω_n in such a way that $u_{(0,T)\times\partial\omega_n}$ admit a strong trace $u^D:=\gamma_{\omega_n}u$ in the L^1 sense, and construct u^ε as solutions to the Cauchy-Dirichlet problem with the smoothed boundary datum $u^{D,\delta}$ converging to u^D as $\delta\to0$.

Indeed, one can represent any open domain Ω in $\mathbb{R}^N$ as a countable union of bounded subdomains Ω_k with C^2 boundary. In each of these subdomains, one considers the parametrization of a neighbourhood of $\partial\Omega_k$ by parameters $\sigma\in\partial\Omega_k$ and $h\in(0,h_{max})$, where $h=\mathrm{dist}(x,\partial\Omega_k)$. A.e. h is a Lebesgue point of the map $h\mapsto u|_{(0,T)\times\Sigma_k^h}$ where $\Sigma_k^h:=\{x\in\Omega_k\,|\,\mathrm{dist}(x,\partial\Omega_k)=h\}$. Thus for every k, one can pick a countable sequence $(\omega_{k,m})_m$ of Lipschitz subdomains of Ω_k such that $\Omega_k=\cup_m\omega_{k,m}$ and u has a *strong* trace (in the L^1 sense) on $(0,T)\times\partial\omega_{k,m}$. We can re-label $\omega_{k,m}$ by a subscript $n\in\mathbb{N}$. From now on, we fix n and write ω for ω_n.

To conclude the proof, combining classical techniques we will construct a vanishing viscosity limit $\tilde u$ which is a Kruzhkov entropy solution of the problem (5.2) in $(0,T)\times\omega$ with the initial condition $\tilde u(0,\cdot)=u(0,\cdot)$ (cf. Remark 2.4 for the issue of time-continuity of local entropy solutions) and the formal boundary condition $\tilde u|_{(0,T)\times\partial\omega}=u^D$, where u^D is the strong trace of u on $(0,T)\times\partial\omega$. Then we will justify the fact that u and $\tilde u$ coincide; notice that at this level, the "rough version" of the Kato inequality (5.3) (see Remark 5.2) is enough to "compare" u and $\tilde u$. Let us provide the details of these arguments.

First, one approximates u^D and $u_0:=u(0,\cdot)$ a.e. on their respective domains by BV functions $u^{D,\delta}$ and u_0^δ. Then one constructs the solutions $\tilde u^{\varepsilon,\delta}$ of (5.4) in $(0,T)\times\omega$ with the corresponding initial and boundary data u_0^δ, $u^{D,\delta}$ using the results of the classical work [11]. As shown in [11], $\tilde u^{\varepsilon,\delta}$ converge, as $\varepsilon\to0$, to an entropy solution $\tilde u^\delta$ of the conservation law (5.2) with the same initial datum u_0^δ and with the same Dirichlet datum $u^{D,\delta}$ understood in the Bardos-LeRoux-Nédélec sense. It remains to obtain $\tilde u=\lim_{\delta\to0}\tilde u^\delta$ and to prove that $\tilde u$ and u do coincide. To this end, we exploit the "rough Kato inequality" of [22] (see Remark 5.2) with test functions of the form $\xi_s(x)\eta(t)$, where $\eta\in\mathcal{D}(0,T)$, $\eta\ge0$, and $(\xi_s)_{s>0}$ is the sequence in $W_0^{1,\infty}(\omega)$ given by $\xi_s=\min\{1,\mathrm{dist}(x,\partial\omega)/s\}$. By a straightforward calculation, at the limit $s\to0$ we find the inequality

$$-\int_0^T\!\!\int_\omega |u-\tilde u^\delta|\,\eta_t \le Const\int_0^T\!\!\int_\omega|u-\tilde u^\delta|\eta - \int_0^T\!\!\int_{\partial\omega}\big(\mathrm{sign}(u^D-\gamma\tilde u^\delta)\big(f(t,x,u^D)-f(t,x,\gamma\tilde u^\delta)\big)\cdot n_{\partial\omega}\big)\,\eta, \quad (5.5)$$

where $n_{\partial\omega}$ is the exterior unit normal vector to $\partial\omega$ and $\gamma\tilde u^\delta$ is the strong trace of the $L^\infty(0,T;BV(\omega))$ function $\tilde u^\delta$. By the result of [11], one has for a.e. (t,x) (with respect to the Hausdorff measure on $(0,T)\times\partial\omega$) the property $(\gamma\tilde u^\delta)(t,x)\in I(t,x,u^{D,\delta}(t,x))$ where

$$I(t,x,v)=\{u\in\mathbb{R}\,|\,\forall k\in[\min\{v,u\},\max\{v,u\}]\ \ \mathrm{sign}(k-v)\big(f(t,x,k)-f(t,x,v)\big)\cdot n_{\partial\omega}\ge0\}.$$

From the definition on $I(t,x,u^{D,\delta})$ and assumption (HA) it is easily seen that the last term in (5.5) is upper bounded by $Const|u^D-u^{D,\delta}|$, which vanishes as $\delta\to0$. Letting $\delta\to0$, using the Gronwall inequality one sees that $\tilde{u}^\delta \to u$ as $\delta \to 0$. Hence one can extract a family $\tilde{u}^{\varepsilon(\delta),\delta}$ of local solutions on (5.4) that converges to u, as $\delta \to 0$. This concludes the proof of the lemma. □

Remark 5.6 *For the one-dimensional stationary problem (i.e., in the context of Proposition 3.3) a simpler construction can be used in the place of the one exploited in the proof of Lemma 5.5. Indeed, it is enough to take, e.g., the function $u|_{\mathbb{R}^l}$ and extend it to $\mathbb{R}$ by setting $\tilde{u}(x) \equiv \gamma^l u = const$ for $x > 0$. Then it is clear that the extension $\tilde{u}$ of u is an entropy solution on $\mathbb{R}$ of the stationary problem $\tilde{u} + \tilde{f}^l(x, \tilde{u}) = \tilde{h}$ with the flux $f^l(x, \cdot)$ extended by $f^l(0, \cdot)$ for $x \geq 0$; also the source term h has to be extended by $\tilde{h}(x) = \gamma^l u = const$ for $x > 0$. Then one can use the classical result of Kruzhkov [22] which guarantees uniqueness of entropy solutions and convergence of vanishing viscosity approximations for the conservation law in the whole space.*

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